A TWO-VARIATE OPERATIONAL CALCULUS FOR BOUNDARY VALUE PROBLEMS *

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Abstract

Using a convolution for the two-point eigenvalue problem $y'' + \lambda^2 y = 0$, $y(0) = y(a) = 0$, one-variate operational calculus for the corresponding boundary value problem is developped. By combining this convolution with the Duhamel convolution, a two-variate convolution is proposed. It is used for a two-variate operational calculus. Applications of this operational calculus to some classical boundary value problems for the heat and the wave equations are given.

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1. Introduction

There exist operational calculi for functions of two or more independent variables following the Mikusinski's scheme. They are based on the Duhamel convolution for functions of several variables. For $n = 2$ this convolution has the form

$$(f * g)(x, y) = \int_0^x \int_0^y f(x - \xi, y - \eta) g(\xi, \eta) d\xi d\eta$$

for functions $f, g \in C(\Delta)$, where $\Delta = [0, \infty) \times [0, \infty)$. However, with this operational calculus however only Caushy problems can be treated. (See Gutterman [1].)

If we are to consider linear boundary value problems, then it is necessary to use new convolutions. Here we will use a convolution intended for boundary value problems.

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2. Operational calculus for the two-point boundary value problem

In [2] explicit convolutions for a class of boundary value problems for $D = \dfrac{d^2}{dx^2}$ are found. Here we are to develop operational calculi connected with the simplest two-point boundary value problem

$$\begin{array}{l} y'' + \lambda^2 y = f(x) \\ y(0) = y(a) = 0 \end{array} . \tag{1}$$

Its solution is the resolvent operator of (1)

$$y = L_{-\lambda^2} f(x) = \frac{1}{\lambda}\int_0^x \sin\lambda(x-\xi) f(\xi) d\xi - \frac{\sin\lambda x}{\lambda\sin\lambda a}\int_0^a \sin\lambda(a-\xi) f(\xi) d\xi.$$

In the case of (1) the general convolution from [2] takes the form

$$(f \star g)(x) = -\frac{1}{2a}\int\limits_0^a \left[\int\limits_x^\xi f(\xi + x - \eta) g(\eta)\, d\eta \right. \tag{2}$$

$$\left. - \int\limits_{-x}^{\xi} f(|\xi - x - \eta|) g(|\eta|)\ \mathrm{sgn}[(\xi - x - \eta)\eta] d\eta \right] d\xi.$$

Having at hand the explicit expression (2), one can check directly that this operation is commutative, associative and the representation

$$L_{-\lambda^2} f(x) = \left\{ \frac{a \sin\lambda x}{\sin\lambda a} \right\} * f(x) \tag{3}$$

for $\lambda \neq 0$ holds. For $\lambda = 0$ we get the right inverse operator L of $\dfrac{d^2}{dt^2}$:

$$Lf(x) = L_0 f(x) = \{x\} * f(x). \tag{4}$$

3. Elements of one-variate operational calculus

Further, by N we denote the subset of $C([0,a])$ of the non-divisiors of 0 for the convolution (2). The set N can easily be characterized in the following way.

LEMMA 1. *A function $f \in C([0,a])$ is a non-divisor of 0 of the convolution algebra $\Big(C([0,a]), *\Big)$ iff*

$$f_n \stackrel{\text{def}}{=} \int_0^a f(\xi) \sin\frac{n\pi\xi}{a} d\xi \neq 0 \tag{5}$$

for all $n \in \mathbb{N}$.

P r o o f. Let us assume that $f_{n_0} = 0$ for some positive integer n_0. Since

$$f * \left\{\sin \frac{n_0\pi x}{a}\right\} = \left(\frac{(-1)^{n_0}}{n_0\pi} \int_0^a f(\xi) \sin \frac{n_0\pi\xi}{a} d\xi\right) \sin \frac{n_0\pi x}{a} = 0,$$

then f is a divisor of 0.

Next we assume that $f_n \neq 0$ for all n. Let $g \in C([0, a])$ be such that $f * g = 0$. Since

$$(f * g) * \left\{\sin \frac{n\pi x}{a}\right\} = \frac{1}{n^2\pi^2} f_n g_n \left\{\sin \frac{n\pi x}{a}\right\} ,$$

then $f_n g_n = 0$ for all n. From (5) it follows $g_n = 0$ for all n. But (see [7])

$$\int_0^a g(\xi) \sin \frac{n\pi\xi}{a} d\xi = 0$$

for all n imply $g = 0$. Hence f is a non-divisor of 0.

Now, following the Mikusinski's scheme, we can build the convolution quotients ring

$$N^{-1}C = \{\frac{f}{g} : g \in N, f \in C\}$$

(see [4], Ch.2, Sec. 3).

Instead of the convolution quotient ring, let us consider the multiplier quotient ring. Although they both are isomorphic, the consideration of multiplier quotients instead of convolution quotients has some advantages. Let us remind the notion of multiplier of the algebra $(C, *)$. An operator $M : C([0, a]) \mapsto C([0, a])$ is said to be a *multiplier of the convolution algebra* $(C, *)$, iff the relation

$$M(f * g) = (Mf) * g$$

holds for arbitrary $f, g \in C([0, a])$.

According to Larsen [5], the multipliers of $(C, *)$ form a commutative ring $\mathcal{M}$ without annihilators. Let us denote by $\mathcal{N}$ the multiplicative set of the non-divisors of 0 of the operator algebra $\mathcal{M}$ with the usual multiplication of operators. $\mathcal{N}$ evidently is nonempty since the identity operator and the convolution operator $\{x\}*$ are non-divisors of 0.

The quotient ring $\mathcal{R} = \mathcal{N}^{-1}\mathcal{M}$ (see [4], Ch.2, Sec.3) is the multipliers quotient ring of the ring $\mathcal{M}$ of the multipliers of the convolution $*$. It consists of fractions A/B, where $A \in \mathcal{M}$, $B \in \mathcal{N}$, i.e.

$$\mathcal{R} = \{\frac{A}{B} : A \in \mathcal{M}, B \in \mathcal{N}\}\,.$$

The function space $C([0, a])$ can also be considered as a subring of $\mathcal{R}$ by means of the correspondence

$$f \mapsto f *\,.$$

In fact, this correspondence is an isomorphic emmbeding of the convolution algebra $(C, *)$ into the multipliers ring $\mathcal{M}$, and hence in $\mathcal{R}$ since $\mathcal{M}$ is considered as a subring of $\mathcal{R}$. For details see [3].

As usual, we denote by 1 the unit element of the ring $\mathcal{R}$. Further, in the operational calculus we introduce the element

$$S = \frac{1}{L}.$$

In the same way as the operator $s = \dfrac{1}{l}$ in the classical Heviside-Mikusinski operational calculus, here the operator S plays a basic role.

Now we establish a fundamental relation between S and f''. In the Mikusinski's operational calculus the basic formula is the relationship between the derivative f' and the product sf:

$$f' = sf - f(0)$$

(cf. [6]), where $f(0)$ is not a constant function, but a so called "numerical operator". In our operational calculus the analogue of this formula of the Mikusinski's operational calculus can be obtained by simplifying the expression Lf''. Since

$$Lf(x) = \int_0^x (x-\xi) f(\xi) d\xi - \frac{x}{a} \int_0^a (a-\xi) f(\xi) d\xi,$$

then

$$Lf'' = f(x) - \left[\left(1 - \frac{x}{a}\right) f(0) + \frac{x}{a} f(a)\right].$$

This equation can be interpreted as an identity in the multiplier's ring $\mathcal{M}$: Lf'' is the product of the multipliers L and $(f'')*$; $\left(1 - \dfrac{x}{a}\right) f(0)$ is the product of the multiplier $\left\{1 - \dfrac{x}{a}\right\}*$ and the "numerical" multiplier $f(0)$; $\dfrac{x}{a} f(a)$ can be considered as the product of the multiplier $L = \{x\}*$ (see (4)) and the numerical multiplier operator $\dfrac{1}{a} f(a)$. Hence

$$Lf'' = f(x) - \left\{1 - \frac{x}{a}\right\} f(0) - L\frac{1}{a} f(a).$$

By multiplication with S we get

$$f'' = Sf - S\left\{1 - \frac{x}{a}\right\} f(0) - \frac{1}{a} f(a). \tag{6}$$

This is the basic formula of the operational calculus for the operator L. It can easily be extended for higher even derivatives of f, e.g.

$$\begin{aligned} f^{(4)} &= Sf'' - S\left\{1 - \frac{x}{a}\right\} f''(0) - \frac{1}{a} f''(a) \qquad (7) \\ &= S^2 f - S\left\{1 - \frac{x}{a}\right\} f''(0) - \frac{1}{a} f''(a) - S^2\left\{1 - \frac{x}{a}\right\} f(0) - S\frac{1}{a} f(a). \end{aligned}$$

As in the Mikusinski's calculus, here the elements $\frac{1}{S-\lambda}$ and $\frac{1}{(S-\lambda)^n}$ of $\mathcal{R}$ can be interpreted as functions of $C([0,a])$.

LEMMA 2.

$$\frac{1}{S+\lambda^2} = \left\{\frac{a\sin\lambda x}{\sin\lambda a}\right\}. \tag{8}$$

P r o o f. Let $f_\lambda = \left\{\frac{a\sin\lambda x}{\sin\lambda a}\right\}$. Then using (6) we get

$$f''_\lambda = Sf_\lambda - S\left\{1-\frac{x}{a}\right\}f_\lambda(0) - \frac{1}{a}f_\lambda(a) = -\lambda^2 f_\lambda.$$

Then $(S+\lambda^2)f_\lambda = 1$ and (8) follows immediately. For the fractions of the form $\frac{1}{S-\lambda^2}$, we get

$$\frac{1}{S-\lambda^2} = \frac{1}{S+(i\lambda)^2} = \left\{\frac{a\sin i\lambda x}{\sin i\lambda a}\right\} = \left\{\frac{a\,\mathrm{sh}\,\lambda x}{\mathrm{sh}\,\lambda a}\right\}. \tag{9}$$

Formulas (8) and (9) can be extended for $\frac{1}{(S\pm\lambda^2)^n}$. For example, formally we get

$$\frac{1}{(S+\lambda^2)^2} = -\frac{1}{2\lambda}\frac{\partial}{\partial\lambda}\left(\frac{1}{S+\lambda^2}\right) = \frac{a^2\cos\lambda a\sin\lambda x - ax\cos\lambda x\sin\lambda a}{2\lambda\sin^2\lambda a}.$$

The formula

$$\frac{1}{(S+\lambda^2)^2} = \frac{a^2\cos\lambda a\sin\lambda x - ax\cos\lambda x\sin\lambda a}{2\lambda\sin^2\lambda a} \tag{10}$$

can be verified either directly or using the convolution (2) for

$$\frac{1}{(S+\lambda^2)^2} = \left\{\frac{1}{S+\lambda^2}\right\} * \left\{\frac{1}{S+\lambda^2}\right\}.$$

4. A two-dimensional convolution

The one-variate operational calculus the elements of which we just developped cannot be used directly for solving boundary value problems for partial differential equations. Having in mind applications to partial differential equations, we develop elements of a two-variate operational calculus. Let us consider the half-strip $\Delta = [0,a]\times[0,\infty)$ and the space $C(\Delta)$ of the continuous functions on Δ. The functions of $C(\Delta)$ will be denoted as $f(x,t)$, $g(x,t)$ ect. The functions of $C(\Delta)$ which do not depend on x or t will be denoted as $f(x)$, $\varphi(t)$ ect. Nevertheless, we consider them as functions of two variables. C_x stands for the subspace of $C(\Delta)$

consisting of functions of the form $F(x,t) = f(x)$ and C_t – for functions of the form $F(x,t) = \varphi(t)$.

THEOREM 1. *The operation*

$$(f \overset{(x,t)}{*} g)(x,t) = -\frac{1}{2a}\int_0^a \int_0^t \left[\int_x^\xi f(x+\xi-\eta, t-\tau)g(\eta,\tau)d\eta \right. \\ \left. - \int_{-x}^\xi f(|\xi-x-\eta|, t-\tau)g(|\eta|,\tau)\mathrm{sgn}\,[(\xi-x-\eta)\eta]d\eta\right] d\tau d\xi \tag{11}$$

is a bilinear, commutative and associative operation in $C(\Delta)$ such that the operators

$$Lf(x,t) = \int_0^x (x-\xi)f(\xi,t)d\xi - \frac{x}{a}\int_0^a (a-\xi)f(\xi,t)d\xi$$

and

$$lf(x,t) = \int_0^t f(x,\tau)d\tau$$

are multipliers of the convolution algebra $\left(C(\Delta), \overset{(x,t)}{*}\right)$.

P r o o f. The bilinearity and the commutativity of (11) are obvious. In order to prove the associativity, it is easy to verify that if $F(x,t) = f(x)\varphi(t)$ and $G(x,t) = g(x)\psi(t)$, then

$$(F \overset{(x,t)}{*} G)(x,t) = (f \overset{x}{*} g)(x)(\varphi \overset{t}{*} \psi)(t),$$

where $(f \overset{x}{*} g)$ is the convolution (2) and

$$(\varphi \overset{t}{*} \psi) = \int_0^t \varphi(t-\tau)\psi(\tau)d\tau$$

is the Duhamel convolution. If $H(x,t) = h(x)\chi(t)$, then

$$(F \overset{(x,t)}{*} G) \overset{(x,t)}{*} H(x,t) = [(f \overset{x}{*} g) \overset{x}{*} h][(\varphi \overset{t}{*} \psi) \overset{t}{*} \chi].$$

But $(f \overset{x}{*} g) \overset{x}{*} h = f \overset{x}{*} (g \overset{x}{*} h)$ and $(\varphi \overset{t}{*} \psi) \overset{t}{*} \chi = \varphi \overset{t}{*} (\psi \overset{t}{*} \chi)$ and hence

$$(F \overset{(x,t)}{*} G) \overset{(x,t)}{*} H = F \overset{(x,t)}{*} (G \overset{(x,t)}{*} H).$$

Then the associativity relation is true for linear combinations of functions of the form considered, and especialy for polynomials of x and t. Since the operation $*$ is continuous in $C(\Delta)$, then the associativity is true for arbitrary $F, G, H \in C(\Delta)$. The fact that the operators L and l are multipliers of the algebra $\left(C(\Delta), \overset{(x,t)}{*}\right)$ reduces to an easy check.

Remark. Neither of the operators L and l can be represented as a convolution operator by the operation $\overset{(x,t)}{*}$. Nevertheless, each of them can be considered as a convolution operator with respect to the one-variable convolutions $f \overset{x}{*} g$ and $\varphi \overset{t}{*} \psi$. $L = \{x\} \overset{x}{*}$ and $l = \{1\} \overset{t}{*}$ since $LF = \{x\} \overset{x}{*} F$ and $lF = \{1\} \overset{t}{*} F$. Their product Ll is the convolution operator $\{x\} \overset{(x,t)}{*}$, i.e.

$$(Ll)\{F(x,t)\} = \{x\} \overset{(x,t)}{*} \{F(x,t)\}. \tag{12}$$

This relation will be useful for the next considerations.

5. Two-variate operational calculus for L and l

Let us denote

$$S = \frac{1}{L}, \quad s = \frac{1}{l}\ .$$

In the same way as before (see (6)), we can express $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial u}{\partial t}$ by Su and su, respectively.

The formula

$$\frac{\partial u}{\partial t} = su - [u(x,0)]_t\ , \tag{13}$$

where $[u(x,0)]_t$ is to be understood as "numerical operator" with respect to the variable t, follows directly from the Mikusinski operational calculus (see [6]).

Now formula (6) takes the form

$$\frac{\partial^2 u}{\partial x^2} = Su - S\left\{(1-\frac{x}{a})\, u(0,t)\right\} - \left[\frac{1}{a}u(a,t)\right]_x , \tag{14}$$

where the subscript x on $\dfrac{1}{a}u(a,t)$ is intended to indicate that it should be considered as numerical multiplier with respect to x and as function with respect to t.

6. Applications

Example 1. Let us consider the general initial-boundary value problem for the heat equation

$$\begin{aligned} &u_t = u_{xx} + F(x,t) \qquad (x,t) \in \Delta \\ &u(x,0) = f(x) \qquad 0 \le x \le a, \\ &u(0,t) = \mu_1(t),\ u(a,t) = \mu_2(t) \qquad t \ge 0, \end{aligned} \tag{15}$$

where Δ is the strip $[0,a]\times[0,\infty)$. Using (13) and (14), the boundary value problem (15) can easily be algebraized:

$$u_t = su - [u(x,0)]_t = su - [f(x)]_t\,,$$

$$u_{xx} = Su - S\left\{(1-\frac{x}{a})\,u(0,t)\right\} - \left[\frac{1}{a}u(a,t)\right]_x = Su - S\left\{(1-\frac{x}{a})\mu_1(t)\right\} - \left[\frac{1}{a}\mu_2(t)\right]_x.$$

Thus, all the problem is reduced to a single algebraic equation in the multipliers quotient ring $\mathcal{R}$:

$$su - [f(x)]_t = Su - S\left\{(1-\frac{x}{a})\mu_1(t)\right\} - \frac{1}{a}[\mu_2(t)]_x + \{F(x,t)\}\,.$$

It is equivalent to

$$(s-S)u = [f(x)]_t - S\left\{(1-\frac{x}{a})\mu_1(t)\right\} - \frac{1}{a}[\mu_2(t)]_x + \{F(x,t)\}\,. \tag{16}$$

Now the problem can be solved by division with $s-S$. To this end one should be sure that $s-S$ is a non-divisor of 0 in $\mathcal{R}$.

THEOREM 2. *The element $s-S$ of $\mathcal{R}$ is a non-divisor of 0 in $\mathcal{R}$.*

P r o o f. Let us assume that for some $a\in\mathcal{R}$ the product $(s-S)a$ is 0, i.e.

$$(s-S)a = 0. \tag{17}$$

The element $a\in\mathcal{R}$ can be represented as $a = P/Q$, where P and Q are multipliers of the convolution algebra $\left(C(\Delta), \overset{(x,t)}{*}\right)$. Then the relation (17) is equivalent to $(L-l)P = 0$. If f is an arbitrary function from $C(\Delta)$, then $(L-l)Pf = 0$. For the sake of simplicity let us denote $Pf = g$. We will show that $(L-l)g = 0$ implies $g = 0$.

Without any loss of generality, we can assume $g(x,0) = 0$. Indeed, $(L-l)g = 0$ imlies $(L-l)lg = 0$, due to the commutation of L and l. Now $(lg)(x,0) = 0$. But the equalities $lg = 0$ and $g = 0$ are equivalent.

Applying the multiplier operators $\left\{\sin\frac{n\pi}{a}x\right\}\overset{x}{*}$ to $(L-l)g = 0$, we get

$$-\left(\frac{a}{n\pi}\right)^2 g_n(t) - l g_n(t) = 0, \tag{18}$$

where

$$g_n(t) = \{g(x,t)\}\overset{x}{*}\left\{\sin\frac{n\pi}{a}x\right\}.$$

The only solution of (18) is $g_n(t)\equiv 0$. Hence

$$\{g(x,t)\}\overset{x}{*}\left\{\sin\frac{n\pi}{a}x\right\} = 0$$

for all n. But

$$\{g(x,t)\} \overset{x}{*} \left\{\sin \frac{n\pi}{a}x\right\} = \left(\frac{(-1)^n}{n\pi}\int_0^a g(\xi,t)\sin\frac{n\pi}{a}\xi d\xi\right)\sin\frac{n\pi}{a}x\,.$$

Since

$$\int_0^a g(\xi,t)\sin\frac{n\pi}{a}\xi d\xi = 0$$

for all n, then $g(x,t)=0$ for all $0\le t<\infty$ (see [7], Ch.3, Sec.13). Thus we proved that $Pf=0$ for each $f\in C(\Delta)$. From this it follows $P=0$ and $a=P/Q=0$. Hence $s-S$ is a non-divisor of 0 in $\mathcal{R}$.

Let us return again to the equation ((16). The formal solution of (16) is

$$u=\frac{1}{s-S}[f(x)]_t+\frac{1}{s-S}\{F(x,t)\}-\frac{S}{s-S}\left\{(1-\frac{x}{a})\mu_1(t)\right\}-\frac{1}{s-S}\frac{1}{a}[\mu_2(t)]_x\,.$$

In order to obtain function-solution we must interpret the various terms in the right-hand side as functions of $C(\Delta)$. Let us begin with the first term $\dfrac{1}{s-S}[f]_t$. Denoting

$$\Omega=\frac{1}{sS(s-S)}=\frac{Ll}{s-S}\,,$$

we can interpret it in the following way. Since $Ll=\{x\}$, then Ω can be considered as a solution to the boundary value problem (15) with $F(x,t)\equiv x$ and $f(x)=0$, $\mu_1(t)=0$, $\mu_2(t)=0$. Such a solution can readily be found, for example by means of Fourier series expansions:

$$\Omega(x,t)=\frac{x}{6}(a^2-x^2)+2\frac{a^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi}{a}x\,\exp\left\{-\left\{\frac{n\pi}{a}\right)^2t\right\}\,.$$

Now we have

$$\frac{1}{s-S}f(x)=sS[f\overset{(x)}{*}\Omega]=\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}[f\overset{(x)}{*}\Omega].$$

Next,

$$\frac{1}{s-S}F(x,t)=\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}[F\overset{(x,t)}{*}\Omega]\,,$$

and so on.

In certain cases, it may be easier to use different solutions. For instance if $F=0$, $\mu_1=0$, $\mu_2=0$, then we can use

$$U=\frac{1}{S(s-S)}=\frac{\partial\Omega}{\partial t}=\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi}{a}x\,\exp\left\{-\left(\frac{\pi}{a}\right)^2t\right\}\,,$$

which is the solution to the heat equation with $f(x)=x$ and $F=0$, $\mu_1=0$, $\mu_2=0$. Then

$$u = \frac{1}{s-S}[f(x)]_t = S[f \overset{(x)}{*} U] = \frac{\partial^2}{\partial x^2}[f \overset{(x)}{*} U].$$

Again, if $f = 0$, $F = 0$, $\mu_1 = 0$ and only $\mu_2 \neq 0$, then

$$u = -\frac{1}{s-S}\frac{1}{a}[\mu_2(t)]_x = \frac{\partial}{\partial t}\frac{1}{a}\mu_2 \overset{(t)}{*} V(x,t) = \frac{1}{a}\frac{\partial}{\partial t}\int_0^t \mu_2(t-\tau)V(x,\tau)d\tau,$$

where

$$V = \frac{1}{s(s-S)} = \frac{\partial^2 \Omega}{\partial x^2} = -x + \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi}{a}x\ \exp\left\{-\left(\frac{n\pi}{a}\right)^2 t\right\},$$

is the solution to the heat equation with $f = 0$, $F = 0$, $\mu_1 = 0$ and $\mu_2 = a$.

The explicit expression of the solution is an extension of the classical Duhamel representation for the solution of non-stationary problems for the heat equation.

Example 2. Let us consider the general initial-boundary value problem for the wave equation

$$\begin{aligned} & u_{tt} = u_{xx} + F(x,t) \qquad (x,t) \in D \\ & u(x,0) = f(x) \qquad 0 \le x \le a, \\ & u_t(x,0) = g(x) \qquad 0 \le x \le a, \\ & u(0,t) = \mu_1(t),\ u(a,t) = \mu_2(t) \qquad t \ge 0. \end{aligned} \tag{19}$$

It can be treated in a similar manner as the problem in Example 1. Here we have

$$\frac{\partial^2 u}{\partial t^2} = s^2 u - s[u(x,0)]_t - [u_t(x,0)]_t$$

and the problem is equivalent to

$$s^2 u - s[f(x)]_t - [g(x)]_t = Su - S\left\{(1-\frac{x}{a})\mu_1(t)\right\} - \frac{1}{a}[\mu_2(t)]_x + \{F(x,t)\},$$

or

$$(s^2 - S)u = s[f(x)]_t + [g(x)]_t - S\left\{(1-\frac{x}{a})\mu_1(t)\right\} - \frac{1}{a}[\mu_2(t)]_x + \{F(x,t)\}.$$

It is easy to verify that $s^2 - S$ is a non-divisor of 0.

Now we find the following formal expression for u:

$$\begin{aligned} u \ = \ & \frac{1}{s^2-S}\{F(x,t)\} + \frac{s}{s^2-S}[f(x)]_t + \frac{1}{s^2-S}[g(x)]_t \\ - \ & \frac{S}{s^2-S}\left\{(1-\frac{x}{a})\mu_1(t)\right\} - \frac{1}{s^2-S}\frac{1}{a}[\mu_2(t)]_x. \end{aligned}$$

Next it remains to interpret the various terms in the right-hand side. Let

$$\Omega = \frac{1}{sS(s^2 - S)} = \frac{Ll}{s^2 - S} .$$

Since $Ll = \{x\}$, then Ω can be interpreted as a solution to the boundary value problem (19) with $F(x,t) \equiv x$ and $f(x) = 0$, $g(x) = 0$, $\mu_1(t) = 0$, $\mu_2(t) = 0$. Such a solution can readily be found, for example by means of Fourier series expansions:

$$\Omega(x,t) = \frac{x}{6}(a^2 - x^2) + 2\frac{a^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi}{a}x\cos\frac{n\pi}{a}t$$

The series for the second derivatives of $\Omega(x,t)$ however does not converge so well as in the case of the heat equation. That is why now we may we consider $\Omega(x,t)$ as a generalized solution of (19).

As in Example 1, we have

$$\frac{1}{s^2 - S}f(x) = sS[f \overset{(x)}{*} \Omega] = \frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}[f \overset{(x)}{*} \Omega],$$

$$\frac{1}{s^2 - S}F(x,t) = \frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}[F \overset{(x,t)}{*} \Omega],$$

etc. The explicit expresson of the solution is an extension of the classical Duhamel representation for the wave equation.

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